Multiplying Fractions

Multiplying fractions challenges students to examine many of the ideas that they have developed about multiplication from their work with whole numbers. The challenge is not one of computation, but rather is one of conceptualization. As Kennedy and Tipps (1997) state,

> Algorithms for multiplication of common fractions are deceptively easy for teachers to teach and for children to use, but their meanings are elusive. Children who are taught rules for performing computation with these algorithms can multiply pairs of fractional numbers with ease. However, if they compute by rules alone, they will understand little of the meanings behind the computation.

Children who know only rules for computing have limited ability to generalize the information to other situations, especially when facing complex problems.

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In the early elementary school years, multiplication with whole numbers is introduced as repeated addition. The repeated-addition model, although a useful link between multiplication and addition, is limited if it is the students' only concept of multiplication. Certain real-life situations represented by multiplication with mixed numbers, common fractions, or decimal fractions cannot easily be interpreted with the repeated-addition model (Graeber and Campbell 1993). Sometimes children have difficulty making sense of such situations.

Teachers should be able to extend multiplication from whole numbers to fractions in a meaningful way. Conceptual models for multiplying fractional numbers, such as the unified approach to multiplying fractions (Ott 1990) and the area model (Graeber and Tanenhaus 1993), can be used to teach the meaning of multiplication of fractions. According to Graeber and Tanenhaus, the area model is a familiar way for students to interpret the multiplication of whole numbers and can be used as a means of conceptualizing the product. This model may help students make sense of the idea of "multiplication making smaller" for such expressions as $9 \times 2/3$ and $1/2 \times 1/10$. These researchers also point out, however, that their expectation about the magnitude of a computational result is likely to interfere with students' ability to make sense of multiplying fractions (Graeber and Tanenhaus 1993). According to the unified approach, repeated addition can be generalized to a product of fractions; in other words, $1/4 \times 3/4$ means one-quarter of a group with a size of three-quarters (the size of one whole group) (Ott 1990). One might wonder how a student would interpret the idea of adding 3/4 to itself 1/4 times. How can a concrete or pictorial representation of the grouping be produced to communicate the abstract idea to students in a meaningful way and allow them to transfer the idea to a variety of problem-solving situations? The multiplication of rational numbers is open to many new and different interpretations compared with whole-number multiplication (Hiebert and Behr 1988). This article attempts to present one alternative model to make the extension of multiplication from whole numbers to fractional numbers meaningful. Before the model is presented and explained, two points are worth mentioning.

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First, students' development of the meaning of multiplication with fractional numbers should emerge from experience with genuine problems (Graeber and Campbell 1993). The reasonable way to start a lesson on the meaning of multiplication with fractional numbers is to allow students to explore situations and make conjectures (NCTM 2000). Mathematical concepts for all operations are rooted in situations and problems. Problem situations should be appropriately designed to provide contexts in which students can solidify their existing knowledge, extend what they know, and further develop generalized ideas about the operation.

Second, ideas that must be generalized are abstract ideas. Educators usually believe that the meanings of mathematical concepts can be communicated to children with the help of manipulatives and pictures. Some mathematical ideas, however, such as the "fractional part of a whole," are abstract concepts that may have no single clear, concrete referent. As such, they are watershed concepts, moving mathematics beyond the concrete world. Students must encounter and reflect on these concepts at a relatively abstract level (Hiebert and Behr 1988). Sometimes, drawing a picture can be thought of as an intermediate step between a mental representation and a physical representation (van Essen and Hamaker 1990).

In the following discussion, real-life problem-solving situations are used to explore the meaning of multiplication of fractions, and pictorial representations are constructed to guide teachers and students to reason about situations, discover patterns, seek ways to validate their own thinking, and convince others that their thinking is correct.

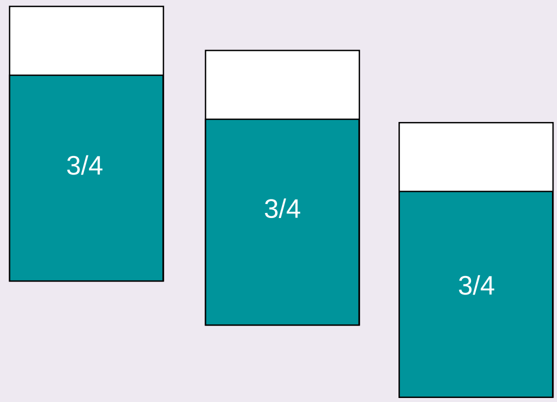
Problem 1

At the supermarket, potatoes were bagged in 3/4-pound bags. Mom bought 3 bags of potatoes. How many pounds of potatoes did Mom buy?

Pictorially, this situation can be represented as shown in **figure 1.** To find the total amount of potatoes, we need to add 3/4 three times. Symbolically, this situation can be represented as 3/4 + 3/4 + 3/4, which is 3 × 3/4. This problem is similar to some of the multiplication situations that students encounter with whole numbers. The link between multiplication and addition is clearly shown. The repeated-addition model offers a satisfying interpretation; that is, the problem deals with three equal-sized groups wherein each group consists of one 3/4-pound bag of potatoes. Situations such as this one can be modeled by the repeated-addition interpretation. We can, however, find other situations that can be represented by multiplication but that do not involve direct repeated-addition situations. Let us look at another problem.

FIGURE 1

Pictorial representation of problem 1



(How many pounds of potatoes did Mom buy?)

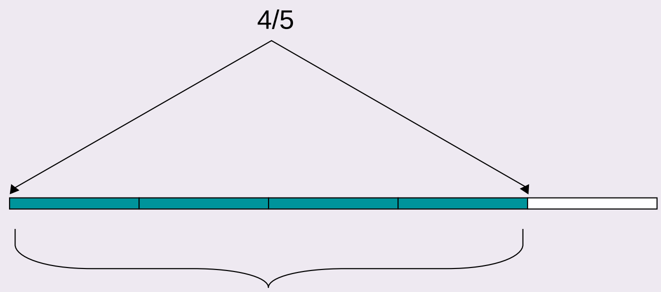
Problem 2

Mrs. Smith has 120 books in her fourth-grade classroom; 4/5 of the books are fiction. How many books are fiction?

Figure 2 shows a pictorial representation of this situation. The bar is used to represent a unit, or a whole, which in this problem is 120 books. Of this whole, 4/5 are fiction books. We divide the whole into five equal parts and shade four parts to represent 4/5 of the whole. In this representation, three components are identified: (1) a whole (unit), (2) a fractional part of the whole, and (3) this fractional part of the whole. For 4/5, the denominator of the fraction, 5, indicates the number of equal-sized groups into which the set of books (the whole) is subdivided: 120 ÷ 5 = 24. Twenty-four books represents one-fifth of the original set of 120 books, or the number of books in one group. The numerator

FIGURE 2

Pictorial representation of problem 2



(How many books are fiction?)

of 4/5, 4, indicates the number of groups being considered: $4 \times 24 = 96$. Ninety-six is the number of books in four groups. For this problem, one sentence, $4/5 \times 120 = 96$, describes both of the operations used. The meaning of the sentence is that multiplication is used to represent the idea of taking a fractional part (4/5) of a whole (120). Problem 3 illustrates a similar situation.

Problem 3

Before the new semester, all the notebooks at the local store are discounted by 1/4. A notebook originally costs \$0.96. How much do you save on one notebook if you buy it today?

In **figure 3,** we again use a bar to represent the original price for a notebook, which is \$0.96. The \$0.96 is considered to be the whole, or the unit, and 1/4 of the whole is discounted. In other words, the purchaser saves 1/4 of the \$0.96 when buying one notebook. The whole \$0.96 is partitioned into sections of equal size. When \$0.96 is subdivided into four equal-sized sections, the mathematical sentence, $0.96 \div 4 = 0.24$, represents the action that has taken place. In this problem, we consider only one section, one-fourth of the original whole, which is \$0.24. The sentence $1/4 \times 0.96 = 0.24$ describes the same procedure. In this situation, we again use multiplication to represent a fractional part of a whole. Problem 4 shows another situation that is similar to those in problems 2 and 3.

FIGURE 3

Pictorial representation of problem 3

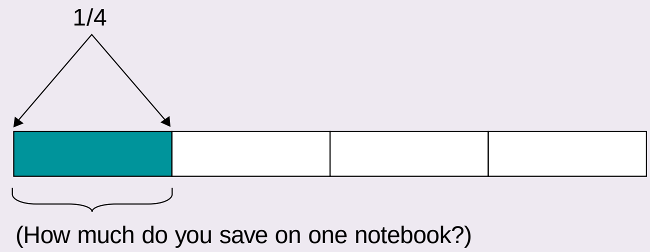
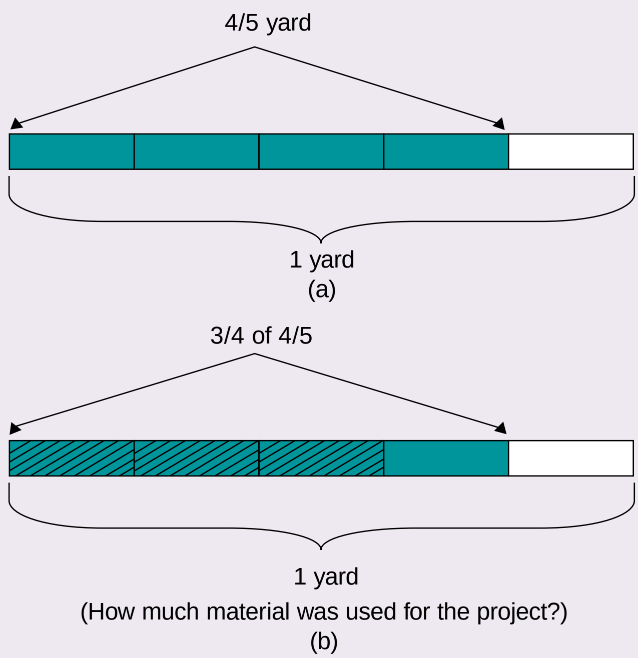


FIGURE 4

Pictorial representation of problem 4



Problem 4

Julie bought 4/5 of a yard of material for her class project. Later, she found that she needed only 3/4 of the material. How much material did Julie use for her project?

The entire bar in **figure 4a** represents a yard of material. The shaded portion is 4/5 of a yard, which now becomes the whole in this problem. We need to find 3/4 of this whole. The whole, 4/5 of a yard, is divided into four equal parts. Each part is 1/5 of a yard. Three parts is 3/5 of a yard. The number sentence $3/4 \times 4/5 = 3/5$ indicates the action that we have taken to find the amount of material used, as shown in **figure 4b.** Again, multiplication is used to consider a fractional part of a whole. This situation is more complex than those in problems 2 and 3 because in this problem, students must realize that the whole is itself part of a unit. The fractions 4/5 and 3/5 refer to two different wholes. As long as the correct whole units are identified, determining what fractional part of which whole to consider becomes obvious.

What about a situation where mixed fractions are involved, such as in this problem:

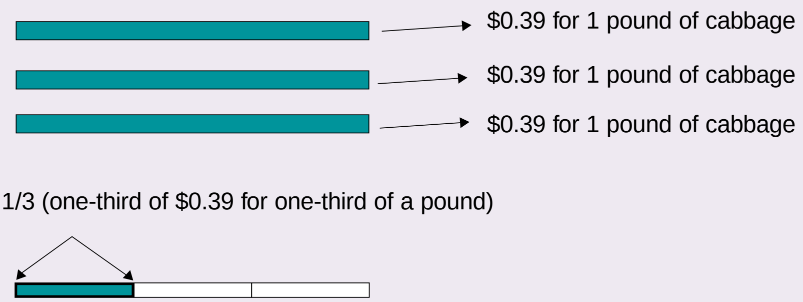
Problem 5

Cabbage costs \$0.39 a pound. Julie bought 3 1/3 pounds of cabbage to prepare her dish. How much did she pay for the cabbage?

Figure 5 shows that this situation calls for a combination of both interpretations, repeated addition *and* part of a whole. The whole is \$0.39. The problem involves three wholes (or equal-sized groups) and a fractional part (1/3) of the whole; the situation can be represented by the number sentence $3\ 1/3 \times \$0.39$. According to the distributive property of multiplication over addition, this sentence is equivalent to $(3 + 1/3) \times 0.39 = 3 \times 0.39 + 1/3 \times 0.39$. The interpretation of multiplication with fractional numbers both as repeated addition

FIGURE 5

Pictorial representation of a problem requiring two interpretations of multiplication



(The total shaded area represents the total cost of the cabbage.)

and as taking a fractional part of a whole provides a satisfactory explanation of this situation.

Conclusion

Problems 2, 3, and 4 illustrate why the repeated-addition model is inadequate for solving certain multiplication problems. In none of the three situations does the pictorial representation present a clear picture of repeated addition. All three situations have one feature in common, that is, each one considers a fractional part of a whole. The whole can be a set of objects or a fractional part of another whole.

Clearly, many multiplicative situations require an appropriate conceptualization of the whole (unit) before the situation can be understood and a solution procedure can be implemented (Hiebert and Behr 1988). In elementary school mathematics, a fraction is typically interpreted as a part-whole relationship, that is, partitioning an object or a set of objects into equal parts. A whole is cut into *n* slices; each slice is encoded as $1/n$; and if we refer to several slices (k), that idea is encoded as k/n. The idea of one whole is a basic feature in this representation.

Although the repeated-addition interpretation of multiplication is ordinarily the first one that students encounter, by the time that operations on whole numbers are expanded to include rational numbers, the meaning of multiplication must also be expanded. The models representing multiplication can be expanded from repeated addition to include a fractional part of a whole. Using real-life contexts, we can categorize many multiplicative situations that students encounter in the middle grades into those that involve repeated addition or those that require taking a fractional part of a whole, depending on the specific situation. To figure out what situation a problem is describing, students must understand the entire problem and be able to develop a coherent representation of it.

References

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